ISRAEL JOURNAL OF MATHEMATICS, Vol. 23, Nos. 3–4, 1976

A FINITELY ADDITIVE VERSION OF KOLMOGOROV'S LAW OF THE ITERATED LOGARITHM†

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ABSTRACT

In their paper *Some finitely additive probability* (to appear in Ann. Probability), Roger A. Purves and William D. Sudderth introduced the measurable strategy idea. In this paper, we first generalize the measurable strategy idea to the more general sigma-fields of subsets of X and prove an important theorem. Then, based on this theorem, we state and prove a finitely additive version of Kolmogorov's law of the iterated logarithm and a finitely additive version of Hartman and Wintner's law of the iterated logarithm in a finitely additive setting.

1. Introduction

Let X be a non-empty set with the discrete topology, $H = X^\infty$ with the product topology, and $F(X)$ be the set of all finitely additive probability measures defined on the class of all subsets of X. As defined by Dubins and Savage in [1], a strategy σ on H is a sequence $\sigma = (\sigma_0, \sigma_1, \sigma_2, \cdots)$, where σ_0 is in $F(X)$ and, for each positive integer n, σ_n is a mapping from X^n to $F(X)$. For each positive integer n, any element $(x_1, x_2, \cdots, x_n)$ in x^n is called a partial history with length n. Let σ be a strategy on H and $p = (x_1, x_2, \cdots, x_n)$ a partial history with length n; then the conditional strategy given the partial history p with respect to the strategy σ is a strategy $\sigma[p] = ((\sigma[p])_0, (\sigma[p])_1, (\sigma[p])_2, \cdots)$ on H defined by (i) $(\sigma[p])_0 = \sigma_n(p) = \sigma_n(x_1, x_2, \cdots, x_n)$, i.e., $(\sigma[p])_0$ is just the finitely additive probability measure $\sigma_n(x_1, x_2, \cdots, x_n)$ and (ii) for each positive integer m, $(\sigma[p])_m$ is a mapping from X^m to $F(X)$ such that $(\sigma[p])_m(x'_1, x'_2, \cdots, x'_m) = \sigma_{n+m}(x_1, x_2, \cdots, x_n, x'_1, x'_2, \cdots, x'_m)$ for all $(x'_1, x'_2, \cdots, x'_m)$ in X^m. In [1] Dubins and

†This research was written with the partial support of the U.S. Army Grant DA-ARO-D-31-124-70-G-102.

Received August 29, 1974 and in revised form September 1, 1975

Savage showed that, for each strategy σ on H, there is a natural finitely additive probability measure induced by the strategy σ defined on the class $\mathcal{K}$ of all clopen (a subset K of H is clopen if K is a closed and open subset of H) subsets of H and they still used σ to denote the induced finitely additive probability measure.

Recently Purves and Sudderth [6] have shown that, for each strategy σ on H, there exists a field $\mathcal{A}(\sigma)$ of subsets of H including all Borel subsets of H such that the finitely additive probability measure σ (induced by the strategy σ and considered by Dubins and Savage in [1]) can be extended from the class $\mathcal{K}$ to the larger field $\mathcal{A}(\sigma)$ with some nice properties (see theors. 2–1, 5–1, and 5–2 of [6]) and they still used σ to denote the new finitely additive probability measure defined on $\mathcal{A}(\sigma)$. Furthermore, they considered measurable strategies in the end of [6] and obtained an important result (see theor. 11–1 of [6]) which shows that the extension they studied agrees with the usual one on all sets of the product sigma-field if assumptions of countable additivity are imposed.

In this paper, we first generalize the measurable strategy idea introduced by Purves and Sudderth to the more general sigma-fields of subsets of X and prove an analogue theorem (Theorem 2.1) to the one in [6]. Then, based on this analogue theorem, we state and prove a finitely additive version of Kolmogorov's law of the iterated logarithm (Theorem 3.1) and a finitely additive version of Hartman and Wintner's law of the iterated logarithm (Theorem 3.2). Finally, we prove a finitely additive version (Theorem 4.1) of theorem 1 of [2] which gives a deeper understanding of the law of the iterated logarithm.

Since, for each strategy σ on H, the field $\mathcal{A}(\sigma)$ of subsets of H includes all Borel subsets of H and σ is a finitely additive probability measure on $\mathcal{A}(\sigma)$, we can consider the triple $(H, \mathcal{A}(\sigma), \sigma)$ as a finitely additive probability space. Hence a definition of integration for the real-valued functions defined on H can be found in [8] and this definition simplifies somewhat in our special setting. Later, we will call a real-valued function Y defined on H σ-integerable if the function Y is integrable with respect to the finitely additive probability space $(H, \mathcal{A}(\sigma), \sigma)$ and $\sigma(Y)$ will be used to denote the integral of Y with respect to $(H, \mathcal{A}(\sigma), \sigma)$.

2. Measurable strategies

Suppose that X is a non-empty set with the discrete topology, $H = X^\infty$ with the product topology, and $(\beta_1, \beta_2, \cdots)$ is a sequence of sigma-fields of subsets of X. For each positive integer m, let $\mathcal{F}^{m,\infty} = \beta_m \times \beta_{m+1} \times \cdots$ be the product sigma-field of subsets of H generated by $(\beta_m, \beta_{m+1}, \cdots)$ and, for each pair (m, n) of positive integers such that $1 \leqq m < n < \infty$, let $\mathcal{F}^{m,n} = \beta_m \times \beta_{m+1} \times \cdots \times \beta_n$ be the product sigma-field of subsets of X^{n-m+1} generated by $(\beta_m, \beta_{m+1}, \cdots, \beta_n)$. A strategy σ on H is said to be measurable with respect to $(\beta_1, \beta_2, \cdots)$ if σ satisfies

i) σ_0 is countably additive when restricted to β_1 and, for each $n = 1, 2, \cdots$, every element $(x_1, x_2, \cdots, x_n) \in X^n$, $\sigma_n(x_1, x_2, \cdots, x_n)$ is countably additive when restricted to β_{n+1}.

ii) For each $n \geqq 1$ and every $A_{n+1} \in \beta_{n+1}$, $\sigma_n(x_1, x_2, \cdots, x_n)(A_{n+1})$ is $\mathcal{F}^{1,n}$-measurable.

Then, by Tulcea's extension theorem (see [5, pp. 162–164]), there is a unique countably additive probability measure ν on $\mathcal{F}^{1,\infty}$ such that $\nu(A) = \sigma(A)$ for every cylinder set A, i.e., $A = B_1 \times B_2 \times \cdots$ where $B_i \in \beta_i$ for all $i \geqq 1$ and for all except finitely many n, $B_n = X$. Let $\mathcal{C} = \mathcal{C}(\mathcal{F}^{1,\infty})$ be the completion of $\mathcal{F}^{1,\infty}$ under ν, then we have the following theorem which generalizes theorem 11–1 of [6].

THEOREM 2.1. *If σ is a measurable strategy with respect to $(\beta_1, \beta_2, \cdots)$, then*

i) *$\mathcal{A}(\sigma)$ contains $\mathcal{C}$*

ii) *σ agrees with ν on $\mathcal{C}$*

where ν is the unique countably additive probability measure on $\mathcal{F}^{1,\infty}$ obtained by Tulcea's extension theorem and $\mathcal{C}$ is the completion of $\mathcal{F}^{1,\infty}$ under ν.

PROOF. The proof is essentially the same as that of theorem 11–1 of [6] (which the present theorem generalizes), but is given here for the sake of completeness and is proved in several rather technical lemmas as follows:

LEMMA 2.1. *Let K be a clopen subset of H and $K \in \mathcal{F}^{1,\infty}$. Then $\sigma(K) = \nu(K)$.*

PROOF. The proof is by induction on the structure of K and has been presented in detail in section 2 of [7]. Therefore we omit it.

An incomplete stop rule t on H is a mapping from H to the set $\{1, 2, 3, \cdots \infty\}$ such that if $h, h' \in H, h'$ agrees with h through the first $t(h)$ coordinates, then $t(h) = t(h')$. A stop rule t is an incomplete stop rule such that $t(h) < \infty$ for all $h \in H$.

LEMMA 2.2. *Let t be a $\mathcal{F}^{1,\infty}$-measurable incomplete stop rule. Then $\sigma(t < \infty) = \nu(t < \infty)$.*

PROOF. Notice that $\sigma(t < \infty) = \sup\{\sigma(t \leqq s) \mid s$ a stop rule on $H\}$ (by corol. 5–3 of [6]) $\geqq \sup\{\sigma(t \leqq n) \mid n \geqq 1\} = \sup\{\nu(t \leqq n) \mid n \geqq 1\}$ (by Lemma 2.1) $= \nu(t < \infty)$ (by the countable additivity of ν on $\mathcal{F}^{1,\infty}$).

To complete the proof, it suffices to show that, for every stop rule τ

$$\sigma(t \leqq \tau) \leqq \sup_n \sigma(t \leqq n) = \sup_n \nu(t \leqq n). \tag{2.1}$$

The proof of (2.1) is by induction on the structure of τ. If τ is structure 0, i.e., τ is a constant, (2.1) is obvious. It remains to check the inductive step.

As defined [1, p. 21], $\tau[x](h) = \tau(xh) - 1$ and $t[x](h) = t(xh) - 1$ for all h in H where $xh = (x, x_1, x_2, \cdots)$ if $h = (x_1, x_2, \cdots) \in H$. As shown in [1, p. 21], for each x, $\tau[x]$ is either a stop rule or identically equal to zero and $\tau[x]$ has smaller structure than that of τ if the structure of τ is larger than zero. Similarly, $t(x)$ is either a $\mathscr{F}^{2,\infty}$-measurable incomplete stop rule or identically equal to zero. Finally, the conditional strategy $\sigma[x]$ is measurable with respect to $(\beta_2, \beta_3, \cdots)$ for each x in X, because σ is measurable with respect to $(\beta_1, \beta_2, \cdots)$. Now let us compute

$$\begin{aligned} \sigma(t \leqq \tau) &= \int \sigma[x]([t \leqq \tau]x) d\sigma_0(x) = \int \sigma[x](t[x] \leqq \tau[x]) d\sigma_0(x) \\ &\leqq \int \sup_n \{\sigma[x](t[x] \leqq n)\} d\sigma_0(x) \end{aligned} \tag{2.2}$$

where $[t \leqq \tau]x = \{h \mid h \in H, t(xh) \leqq \tau(xh)\}$ and the inequality follows from the inductive assumption.

Let $\varepsilon > 0$ and, for each $x \in X$, $N(x) = \min\{k : (\sigma[x](t[x] \leqq k) \geqq [\sup_n \sigma[x](t[x] \leqq n)] - \varepsilon\}$, and let $M(h) = N(x_1) + 1$ for $h = (x_1, x_2, \cdots) \in H$. Then, by (2.2),

$$\begin{aligned} \sigma(t \leqq \tau) &\leqq \int \sigma[x](t[x] \leqq N(x)) d\sigma_0(x) + \varepsilon = \int \sigma[x]([t \leqq M]x) d\sigma_0(x) + \varepsilon \\ &= \sigma(t \leqq M) + \varepsilon = \nu(t \leqq M) + \varepsilon. \end{aligned} \tag{2.3}$$

The last step, which follows from Lemma 2.1, requires that M be $\mathscr{F}^{1,\infty}$-measurable. This will follow easily from the β_1-measurability of the function $x \to \sigma[x](Ax)$, where A is $\mathscr{F}^{1,\infty}$-measurable and has finite structure. The quantity $\sigma[x](Ax)$ can be evaluated in a natural way as an iterated integral involving finitely additive extensions of the countably additive $\sigma[p]$'s (see [1, p. 13]). A little reflection shows that the iterated integral has the same value as the usual Lebesgue integral. The β_1-measurability of "$x \to \sigma[x](Ax)$" then follows by the standard arguments.

Since M is $\mathscr{F}^{1,\infty}$-measurable and ν is countably additive, there exists a positive integer n such that $\nu(M \leqq n) \geqq 1 - \varepsilon$. So, by (2.3), $\sigma(t \leqq \tau) \leqq \nu(t \leqq M) + \varepsilon \leqq$

$\nu(t \leqq n) + 2\varepsilon = \sigma(t \leqq n) + 2\varepsilon$ (the last equation is implied by Lemma 2.1). Since ε is arbitrary, (2.1) is now proved.

Let $\mathcal{D}$ be the collection of all $\mathcal{F}^{1,\infty}$-measurable incomplete stop rules t. For $A \subseteq H$, let

$$\nu^*(A) = \inf\{\nu(t < \infty) \mid t \in \mathcal{D}, A \subseteq [t < \infty]\} \quad \text{and}$$

$$\nu_*(A) = \sup\{\nu(t = \infty) \mid t \in \mathcal{D}, A \supseteq [t = \infty]\}.$$

Then it is easy to see that $\nu_*(A) = 1 - \nu^*(A^c)$ for all $A \subseteq H$.

LEMMA 2.3. *Let $\mathcal{C}' = \{A \mid A \subseteq H, \nu^*(A) = \nu_*(A)\}$, then the collections $\mathcal{C}$ and $\mathcal{C}'$ coincide. Also, ν^* restricted to $\mathcal{C}$ is the completion of ν and, in particular, ν^* is countably additive on $\mathcal{C}$.*

PROOF. By checking in order that $\mathcal{C}'$ is closed under the taking of complements, finite unions, and countable increasing unions, it is easy to see that $\mathcal{C}'$ is a sigma-field.

Now let A be a cylinder set in $\mathcal{F}^{1,\infty}$. Then there is a positive integer n and a set $B \subseteq X^n$ such that $A = \{(x_1, x_2, \cdots, x_n, \cdots) \mid (x_1, x_2, \cdots, x_n) \in B\}$. Let $t(h) = \infty$ or n according as $h \notin A$ or $h \in A$; and $\tau(h) = \infty$ or n according as $h \in A$ or $h \notin A$. Then $t, \tau \in \mathcal{D}$ and $[t < \infty] = A = [\tau = \infty]$. Thus $A \in \mathcal{C}'$ and $\mathcal{C}' \supseteq \mathcal{F}^{1,\infty}$.

To see $\mathcal{C}' \subseteq \mathcal{C}$, let $A \in \mathcal{C}'$. Write O for sets of the form $[t < \infty]$ and C for sets of the form $[t = \infty]$ when $t \in \mathcal{D}$. Then there exist sets O_n and C_n such that the O_n are decreasing, the C_n are increasing, $O_n \supseteq A \supseteq C_n$, $\nu(O_n) \to \nu^*(A)$, and $\nu(O_n - C_n) \to 0$. Thus $\bigcup_{n=1}^{\infty} C_n \subseteq A$, $A - \bigcup_{n=1}^{\infty} C_n \subseteq \bigcap_{n=1}^{\infty} O_n - \bigcup_{n=1}^{\infty} C_n$, and $\nu(\bigcap_{n=1}^{\infty} O_n - \bigcup_{n=1}^{\infty} C_n) = 0$. Thus A differs from $\bigcup_{n=1}^{\infty} C_n$ by a subset of a $\mathcal{F}^{1,\infty}$-measurable set which is ν-null and, hence, $A \in \mathcal{C}$. Notice also that $\nu^*(A) = \nu(\bigcup_{n=1}^{\infty} C_n)$. Hence ν^* agrees with the completion of ν on $\mathcal{C}'$. But $\mathcal{C}'$ is clearly complete for ν^* and so is complete for ν. Therefore $\mathcal{C} = \mathcal{C}'$.

As in Section 1, $\mathcal{K}$ denotes the class of all clopen subsets of H, σ denotes the finitely additive probability measure defined on $\mathcal{K}$ (induced by the strategy σ). For each open subset O of H, Let $\sigma(O) = \sup\{\sigma(K) \mid K \in \mathcal{K}\ K \subseteq O\}$ and, for each closed subset C of H, let $\sigma(C) = \inf(\sigma(K) \mid K \in \mathcal{K}, K \supseteq C\}$. For each subset A of H, let $\sigma^*(A) = \inf\{\sigma(O) \mid O$ is open and $A \subseteq O\}$ and let $\sigma_*(A) = \sup\{\sigma(C) \mid C$ is closed and $C \subseteq A\}$. Let $\mathcal{A}(\sigma) = \{A \mid A \subseteq H, \sigma^*(A) = \sigma_*(A)\}$ and if $A \in \mathcal{A}(\sigma)$, we write $\sigma(A)$ for $\sigma^*(A)$ $(= \sigma_*(A))$.

The next lemma finishes the proof of Theorem 2.1.

LEMMA 2.4. *For every $A \subseteq H$, $\nu^*(A) \geqq \sigma^*(A) \geqq \sigma_*(A) \geqq \nu_*(A)$. Hence, $\mathcal{A}(\sigma) \supseteq \mathcal{C}$ and σ^* agrees with ν^* on $\mathcal{C}$ (σ agrees with ν on $\mathcal{C}$).*

PROOF. By Lemma 2.2.

3. Finitely additive versions of the law of the iterated logarithm

In this section, we, first, state and prove a finitely additive version of Kolmogorov's law of the iterated logarithm (based on Theorem 2.1) and then state a finitely additive version of Hartman and Wintner's law of the iterated logarithm without the proof. All proofs are valid for a countably additive setting if we consider the problems in a coordinate representation process. Therefore, the results in this section are generalizations of the classical results in a coordinate representation process. We start with the following definitions.

A strategy σ on H is an independent strategy if there exists a sequence $\{\gamma_n\}$ in $F(X)$ such that $\sigma_0 = \gamma_1$ and, for each positive integer n and all n-tuples $p = (x_1, x_2, \cdots, x_n)$ in X^n, $\sigma_n(p) = \gamma_{n+1}$. $\sigma = \gamma_1 \times \gamma_2 \times \cdots$ is written for such a strategy. A strategy σ on H is an independent and identically distributed strategy if there is a finitely additive probability measure γ in $F(X)$ such that $\sigma_0 = \gamma$ and, for each positive integer n and every element $(x_1, x_2, \cdots, x_n)$ in X^n, $\sigma_n(x_1, x_2, \cdots, x_n) = \gamma$. We will write $\sigma = \gamma \times \gamma \times \cdots$ for such a strategy.

A sequence $\{Y_n\}$ of real-valued functions defined on H is called a sequence of coordinate mappings on H if, for each $n = 1, 2, \cdots$, Y_n depends only on the nth coordinate. A sequence $\{Y_n\}$ of real-valued functions defined on H is called a sequence of identical and coordinate mappings if $\{Y_n\}$ is a sequence of coordinate mappings on H and $Y_n(h) = Y_m(h)$ whenever $h = (x_1, x_2, \cdots, x_n, \cdots, x_m, \cdots)$ in H and $x_n = x_m$ for all $m = 1, 2, \cdots$, $n = 1, 2, \cdots$.

THEOREM 3.1. *Suppose that $\sigma = \gamma_1 \times \gamma_2 \times \cdots$ is an independent strategy on H and $\{Y_n\}$ is a sequence of coordinate mappings defined on H such that $\sigma(Y_n) = 0$ and $\sigma(Y_n^2) < \infty$ for all $n \geqq 1$. For each $n \geqq 1$, let $a_n^2 = \sum_{j=1}^n \sigma(|Y_j - \sigma(Y_j)|^2) = \sum_{j=1}^n \sigma(Y_j^2)$, $b_n = 2 \log\log a_n^2$ if $a_n^2 \geqq e^e$, $b_n = 2$ if $a_n^2 \leqq e^e$. Suppose that* (i) *$\lim_{n\to\infty} a_n^2 = \infty$ and* (ii) *for each $n \geqq 1$, there exists a positive constant K_n such that $|Y_n| \leqq K_n a_n b_n^{-1}$ and $\lim_{n\to\infty} K_n = 0$. Then, we have*

$$\text{i)}\quad \sigma\left(\left[h \,\middle|\, \limsup_{n\to\infty}\left\{\sum_{j=1}^n Y_j(h)/a_n b_n\right\} = 1\right]\right) = 1.$$

$$\text{ii)}\quad \sigma\left(\left[h \,\middle|\, \liminf_{n\to\infty}\left\{\sum_{j=1}^n Y_j(h)/a_n b_n\right\} = -1\right]\right) = 1.$$

PROOF. It suffices to prove (i) and it is proved in the following constructive steps:

Since, for each $n \geqq 1$, $\sigma(Y_n^2) < \infty$, there exists a positive integer $M_n \geqq n$ such that (a) $\sigma([h \mid |Y_n(h)| > M_n]) \leqq n^{-2}$ and (b) if the real-valued function Z_n is defined by $Z_n(h) = -M_n$ if $Y_n(h) \leqq -M_n$, $Z_n(h) = M_n$ if $Y_n(h) \geqq M_n$ and

$Z_n(h) = K + j2^{-M_n}$ if $K + j2^{-M_n} \leqq Y_n(h) < K + (j+1)2^{-M_n}$ where $K = -M_n, -M_n + 1, \cdots, 0, 1, \cdots, M_n - 1$ and $j = 0, 1, 2, \cdots, 2^{M_n} - 1$, then Z_n has the following two properties: (i) $|\sigma(Z_n) - \sigma(Y_n)| = |\sigma(Z_n)| \leqq n^{-2}$ and (ii) $|\sigma(|Z_n - \sigma(Z_n)|^2) - \sigma(Y_n^2)| \leqq n^{-2}$.

Since, for each $n \geqq 1$, Y_n depends only on the nth coordinate, Z_n depends only on the nth coordinate. Hence the set X can be decomposed into finitely many disjoint subsets of X by Z_n; let P_n be the partition of X by Z_n. For each $n \geqq 1$, let β_n be the sigma-field of subsets of X generated by the class P_n. Since P_n is a finite set, β_n is finite and γ_n is countably additive when restricted to β_n. Since $\sigma = \gamma_1 \times \gamma_2 \times \cdots$, it is obvious that, for each $n \geqq 1$ and every $A_{n+1} \in \beta_{n+1}$, $\sigma_n(x_1, x_2, \cdots, x_n)(A_{n+1})$ is $\mathscr{F}^{1,n}$-measurable where $\mathscr{F}^{1,n} = \beta_1 \times \beta_2 \times \cdots \times \beta_n$ is the product sigma-field of subsets of X^n generated by $(\beta_1, \beta_2, \cdots, \beta_n)$. Now, by Theorem 2.1, there exists a countably additive probability measure ν on $\mathscr{F}^{1,\infty} = \beta_1 \times \beta_2 \times \cdots$ such that $\mathscr{A}(\sigma) \supseteq \mathscr{C} = \mathscr{C}(\mathscr{F}^{1,\infty})$ (the completion of $\mathscr{F}^{1,\infty}$ under ν) and for all $A \in \mathscr{C}$, $\sigma(A) = \nu(A)$.

It is obvious that, for each $n \geqq 1$, Z_n is $\mathscr{F}^{1,\infty}$-measurable and $\int_H Z_n(h)d\nu(h) = \nu(Z_n) = \sigma(Z_n)$,

$$\int_H [Z_n(h) - \nu(Z_n)]^2 d\nu(h) = \nu(|Z_n - \nu(Z_n)|^2) = \sigma(|Z_n - \sigma(Z_n)|^2).$$

Now, for each $n \geqq 1$, let

$$d_n^2 = \sum_{j=1}^n \sigma(|Z_j - \sigma(Z_j)|^2) = \sum_{j=1}^n \nu(|Z_j - \nu(Z_j)|^2)$$

and $e_n^2 = 2 \log\log d_n^2$ if $d_n^2 \geqq e^e$, $e_n^2 = 2$ if $d_n^2 \leqq e^e$. Since $|\sigma(|Z_n - \sigma(Z_n)|^2) - \sigma(|Y_n - \sigma(Y_n)|^2)| \leqq n^{-2}$ for all $n \geqq 1$ and $\lim_{n\to\infty} a_n^2 = \infty$, $\lim_{n\to\infty}\{d_n^2/a_n^2\} = 1$. Since $|Z_n| \leqq |Y_n| + 2^{-M_n} \leqq |Y_n| + 2^{-n}$ (recall that $M_n \geqq n$), $\lim_{n\to\infty}\{d_n^2/a_n^2\} = 1$, and $\lim_{n\to\infty} a_n^2 = \infty$, we can find a sequence $\{L_n\}$ of positive real numbers such that $|Z_n| \leqq L_n d_n e_n^{-1}$ for all $n \geqq 1$ and $\lim_{n\to\infty} L_n = 0$. Now, applying Kolmogorov's law of the iterated logarithm to the probability space $\{H, \mathscr{C}, \nu\}$ and the random variables $\{Z_n\}$, we have

$$\nu\left(\left[h \,\middle|\, \limsup_{n\to\infty}\left\{d_n^{-1}e_n^{-1}\sum_{j=1}^n [Z_j(h) - \nu(Z_j)]\right\} = 1\right]\right) = 1. \tag{3.1}$$

Notice that, for each $n \geqq 1$,

$$\sigma([h \,||\, Y_n(h) - Z_n(h)| > 2^{-n}]) \leqq \sigma([h \,||\, Y_n(h)| > M_n]) \leqq n^{-2}(n \leqq M_n).$$

Hence $\Sigma_{n=1}^\infty \sigma([h \,||\, Y_n(h) - Z_n(h)| > 2^{-n}]) < \infty$. By theorem 7–1 of [6] (a finitely additive version of the Borel Cantelli lemma), we have $\sigma(G_1)=1$ where $G_1=[h \,\|\, Y_n(h)-Z_n(h)|>2^{-n} \text{ i.o.}]^c$. Let

$$G_2=\left[h \,\middle|\, \limsup_{n\to\infty}\left\{d_n^{-1}e_n^{-1}\sum_{j=1}^{n}[Z_j(h)-\nu(Z_j)]\right\}=1\right],$$

then $\sigma(G_2)=\sigma(G_1)=\sigma(G_1\cap G_2)=1$.

Now, it is enough to show that

$$G_1\cap G_2\subseteq\left[h \,\middle|\, \limsup_{n\to\infty}\left\{a_n^{-1}b_n^{-1}\sum_{j=1}^{n}Y_j(h)\right\}=1\right]. \tag{3.2}$$

First notice that if $h\in G_1$, then

$$\begin{aligned}\left|\sum_{j=1}^{n}Y_j(h)-\sum_{j=1}^{n}[Z_j(h)-\sigma(Z_j)]\right| &\leqq \left|\sum_{j=1}^{K(h)}[Y_j(h)-Z_j(h)+\sigma(Z_j)]\right| \\ &\quad+\sum_{j=K(h)+1}^{n}|Y_j(h)-Z_j(h)|+\sum_{j=K(h)+1}^{n}|\sigma(Z_j)| \\ &\leqq \left|\sum_{j=1}^{K(h)}[Y_j(h)-Z_j(h)+\sigma(Z_j)]\right|+\sum_{j=K(h)+1}^{n}(2^{-j}+j^{-2}),\end{aligned}$$

where $K(h)$ is the last time such that $h\in[h' \,\|\, Y_n(h')-Z_n(h')|>2^{-n}]$. Since $\lim_{n\to\infty}\{a_nb_n\}=\infty$ and $\lim_{n\to\infty}\{d_ne_n/a_nb_n\}=1$,

$$\begin{aligned}\limsup_{n\to\infty}\left\{a_n^{-1}b_n^{-1}\cdot\sum_{j=1}^{n}Y_j(h)\right\}&=\limsup_{n\to\infty}\left\{a_n^{-1}b_n^{-1}\sum_{j=1}^{n}[Z_j(h)-\sigma(Z_j)]\right\}\\&=\limsup_{n\to\infty}\left\{d_n^{-1}e_n^{-1}\sum_{j=1}^{n}[Z_j(h)-\nu(Z_j)]\right\}\end{aligned}$$

if $h\in G_1$. Next, if $h\in G_2$, then $\limsup_{n\to\infty}\{d_n^{-1}e_n^{-1}\Sigma_{j=1}^{n}[Z_j(h)-\nu(Z_j)]\}=1$. Therefore, if $h\in G_1\cap G_2$, then $\limsup_{n\to\infty}\{a_n^{-1}b_n^{-1}\cdot\Sigma_{j=1}^{n}Y_j(h)\}=1$. Hence (3.2) holds and the proof of Theorem 3.1 now is complete.

THEOREM 3.2. *Suppose that $\sigma=\gamma\times\gamma\times\cdots$ is an independent and identically distributed strategy on H and $\{Y_n\}$ is a sequence of identical, coordinate mappings defined on H. Suppose that $\sigma(Y_1)=0$ and $\sigma(Y_1^2)=1$. Then, we have*

i) $$\sigma\left(\left[h \,\middle|\, \limsup_{n\to\infty}\frac{1}{\sqrt{2n\log\log n}}\sum_{j=1}^{n}Y_j(h)=1\right]\right)=1.$$

ii) $$\sigma\left(\left[h \,\middle|\, \liminf_{n\to\infty}\frac{1}{\sqrt{2n\log\log n}}\sum_{j=1}^{n}Y_j(h)=-1\right]\right)=1.$$

PROOF. The proof is essentially the same as the one given in [3], i.e., reduce the sequence $\{Y_n\}$ to a new sequence $\{Z_n\}$ such that $\{Z_n\}$ satisfies the conditions stated in Theorem 3.1. Next, show that

i) $\lim_{n\to\infty} \frac{1}{\sqrt{2n \log\log n}} \sum_{j=1}^{n} \sigma(Z_j) = 0,$

ii) $\lim_{n\to\infty} \left[\sum_{j=1}^{n} \sigma(Z_j - \sigma(Z_j))^2 \Big/ \sqrt{2n \log\log n} \right] = 1,$ and

iii) $\sigma([h \mid Y_n(h) \neq Z_n(h) \text{ i.o.}]) = 0.$

Therefore, the details of the proof are omitted.

4. A further remark

In [2], Freedman obtained the following result: "Suppose that $Y_1, Y_2, \cdots$ are uniformly bounded i.i.d. random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ such that $E(Y_1) = 0$. Then $E\{\exp(tS^2)\} = \int_\Omega \exp(tS^2(w))dP(w) < \infty$ for all $0 < t < \infty$, where $S = \sup_{n\geqq 3}\{\sum_{j=1}^{n} Y_j / \sqrt{n \log\log n}\}$". This result gives us a deeper understanding of the law of the iterated logarithm. In this section, we will show that the Freedman result holds in our finitely additive setting.

LEMMA 4.1. *Suppose that $\sigma = \gamma \times \gamma \times \gamma \times \cdots$ is an independent, identically distributed strategy on H and $\{Y_n\}$ is a sequence of identical, coordinate mappings defined on H. Suppose that $\sigma(Y_1) = 0$, $\sigma(Y_1^2) = a^2 > 0$, $|Y_1| \leqq K < \infty$ (K is a positive constant), and there exists a positive real number ε_0 such that $\sigma(Y_1 \geqq 0) \geqq \varepsilon_0 > 0$. Then,* $\inf_{n\geqq 1} \sigma([h \mid \sum_{j=1}^{n} Y_j(h) \geqq 0]) = c > 0$.

PROOF. Without loss of generality, we can and do assume that $K = 1$. For each $n \geqq 1$, choose a positive integer $M_n \geqq n$ such that if the real-valued function Z_n defined by $Z_n(h) = l + j2^{-M_n}$ if $l + j2^{-M_n} \leqq Y_n(h) < l + (j+1)2^{-M_n}$, $Z_n(h) = 1$ if $Y_n(h) = 1$ where $l = -1$, 0 and $j = 0, 1, 2, \cdots, 2^{M_n} - 1$, then $1 \geqq \sigma(|Z_n - \sigma(Z_n)|^2) \geqq (1 - 2n^{-2})a^2$.

Now, as in Theorem 3.1, we obtain a sequence $\{\beta_n\}$ of sigma-fields of subsets of X by the sequence $\{Z_n\}$. Let $\mathcal{F}^{1,\infty} = \beta_1 \times \beta_2 \times \cdots$, ν be the uniquely countably additive probability measure defined on $\mathcal{F}^{1,\infty}$ (obtained by Tulcea's extension theorem). Notice that $Z_n \leqq Y_n \leqq Z_n + 2^{-n}(M_n \geqq n)$, hence $\sigma(Z_n) \leqq \sigma(Y_n) \leqq \sigma(Z_n) + 2^{-n}$, i.e., $0 \geqq \sigma(Z_n) \geqq -2^{-n}$. Now by the central limit theorem,

$$\lim_{n\to\infty} \nu\left(\left[h \,\middle|\, \left[\sum_{j=1}^{n} Z_j(h) - \sum_{j=1}^{n} \nu(Z_j)\right] \Big/ \sqrt{\sum_{j=1}^{n} (|Z_j - \nu(Z_j)|^2)} \geqq 1\right)\right.$$

$$= \int_1^\infty \frac{1}{\sqrt{2\pi}} \exp(-t^2/2)dt = 2c_1 > 0.$$

Since $0 \geqq \sum_{j=1}^{n} \nu(Z_j) > -1$ for all $n \geqq 1$,

$$\liminf_{n\to\infty} \nu\left(\left[h \,\middle|\, \left[\sum_{j=1}^{n} Z_j(h) \middle/ \sqrt{\sum_{j=1}^{n} \nu(|Z_j - \nu(Z_j)|^2)} \geqq 0\right]\right]\right)$$

$$\geqq \lim_{n\to\infty} \nu\left(\left[k \,\middle|\, \left[\sum_{j=1}^{n} Z_j(h) - \sum_{j=1}^{n} \nu(Z_j)\right] \middle/ \sqrt{\sum_{j=1}^{n} \nu(|Z_j - \nu(Z_j)|^2)} \geqq 1\right]\right) = 2c_1 > 0.$$

Therefore, there exists a positive integer N such that, if $n > N$, $\nu([h \mid \sum_{j=1}^{n} Z_j(h) \geqq 0]) \geqq c_1 > 0$. But $[h \mid \sum_{j=1}^{n} Z_j(h) \geqq 0] \subseteq [h \mid \sum_{j=1}^{n} Y_j(h) \geqq 0]$, hence $\sigma([h \mid \sum_{j=1}^{n} Y_j(h) \geqq 0]) \geqq c_1 > 0$ if $n > N$.

For $1 \leqq n \leqq N$,

$$\sigma\left(\left[h \,\middle|\, \sum_{j=1}^{n} Y_j(h) \geqq 0\right]\right) \geqq \sigma([h \mid Y_j(h) \geqq 0, j = 1, 2, \cdots, n])$$

$$= \prod_{j=1}^{n} \sigma([h \mid Y_j(h) \geqq 0]) \geqq \varepsilon_0^n \geqq \varepsilon_0^N > 0.$$

Therefore $\inf_{n \geqq 1} \sigma([h \mid \sum_{j=1}^{n} Y_j(h) \geqq 0]) \geqq \min\{\varepsilon_0^N, c_1\} > 0$.

Lemma 4.2. *Suppose that σ and $\{Y_n\}$ are as defined in Lemma 4.1. Let, for each $n \geqq 1$, Z_n be the real-valued function defined on H by $Z_n(h) = l + (j+1)2^{-n}$ if $l + j2^{-n} < Y_n(h) \leqq l + (j+1)2^{-n}$ and $Z_n(h) = 2^{-n} - 1$ if $Y_n(h) - 1$ where $l = 0, -1$ and $j = 0, 1, 2, \cdots, 2^n - 1$. Then*

$$\inf_{m \geqq 1} \inf_{n \geqq m} \sigma\left(\left[h \,\middle|\, \sum_{j=m}^{n} Z_j(h) \geqq 0\right]\right) = d > 0.$$

Proof. Since $[h \mid \sum_{j=m}^{n} Z_j(h) \geqq 0] \supseteq [h \mid \sum_{j=m}^{n} Y_j(h) \geqq 0]$, $\sigma = \gamma \times \gamma \times \cdots$ is an independent, identically distributed strategy, and $\{Y_n\}$ is a sequence of identical, coordinate mappings defined on H, by Lemma 4.1, Lemma 4.2 is obvious.

Lemma 4.3. *Suppose that σ, $\{Y_n\}$, and $\{Z_n\}$ are as defined in Lemma 4.2. Then $\sigma(e^{tZ_1}) \geqq \sigma(e^{tZ_n})$ for all $0 \leqq t < \infty$ and $n \geqq 1$.*

Proof. The proof is straightforward and is omitted.

Lemma 4.4. *Suppose that σ, $\{Y_n\}$, and $\{Z_n\}$ are as defined in Lemma 4.3 and suppose that $T = \sup_{n \geqq 3}\{\sum_{j=1}^{n} Z_j / \sqrt{n \log\log n}\}$. Then, for all $0 < t < \infty$, $\sigma(e^{tT^2}) < \infty$.*

Proof. As in Lemma 4.1, using $\{Z_n\}$, we can obtain a sequence $\{\beta_n\}$ of sigma-fields of subsets of X and a unique countably additive probability measure ν on the product sigma field $\mathscr{F}^{1,\infty} = \beta_1 \times \beta_2 \times \cdots$. Since T^2 is $\mathscr{F}^{1,\infty}$-measurable, it is sufficient to show that $\int_H e^{tT^2(h)} d\nu(h) = \nu(e^{tT^2}) < \infty$ for all $t < 0$.

Notice that

$$T=\sup_{n\geqq 3}\left\{\sum_{j=1}^{n} Z_j/\sqrt{n\log\log n}\right\}\geqq\left\{\sum_{j=1}^{3} Z_j/\sqrt{3\log\log 3}\right\}\geqq -3/\sqrt{3\log\log 3},$$

hence $[h\mid T(h)<s]=\phi$ if $s<-3/\sqrt{3\log\log 3}$. Plainly for $1<L<\infty$, $\nu(e^{tT^2})\leqq L+\int_L^\infty \nu([h\mid e^{tT^2(h)}>w])dw$. Now let $I^+(L)=\int_L^\infty \nu([h\mid T(h)>t^{-\frac{1}{2}}(\log w)^{\frac{1}{2}}])dw$ and $I^-(L)=\int_L^\infty \nu([h\mid T(h)<-t^{-\frac{1}{2}}(\log w)^{\frac{1}{2}}])dw$. It is obvious that $I^-(L)<\infty$ for all $L>0$, so it suffices to show that, for L large, $I^+(L)<\infty$. The remainder of the argument for $I^+(L)<\infty$ for large L is essentially the same as the one given in [2] except that we have to notice that

$$\nu\left(\left[h\;\middle|\;\sum_{j=1}^{n} Z_j(h)>y\right]\right)\leqq e^{-ty}\nu\left(\exp\left(t\sum_{j=1}^{n} Z_j\right)\right)\quad (t>0,y>0)=e^{-ty}\prod_{j=1}^{n}\nu(e^{tZ_j})$$
$$\leqq e^{-ty}[\nu(e^{tZ_1})]^n$$

(by Lemma 4.3). Therefore, we omit the detail.

THEOREM 4.1. *Suppose that $\sigma=\gamma\times\gamma\times\cdots$ is an independent, identically distributed strategy on H and $\{Y_n\}$ is a sequence of identical, coordinate mappings defined on H such that $|Y_1|\leqq K<\infty$ (K is a positive constant) and $\sigma(Y_1)=0$. Then $\sigma(e^{tS^2})<\infty$ for all $0\leqq t<\infty$, where $S=\sup_{n\geqq 3}\{\sum_{j=1}^n Y_j/\sqrt{n\log\log n}\}$.*

PROOF. Without loss of generality, we can and do assume that $K=1$. Now, if $\sigma(Y_1^2)=0$ or $\sigma(Y_1^2)>0$ but $\sigma(Y_1<0)=1$, then Theorem 4.1 is obvious. So we assume that $\sigma(Y_1^2)=a^2>0$ and $\sigma(Y_1\geqq 0)=\varepsilon_0>0$. Now, for each $n\geqq 1$, let $Z_n(h)=l+(j+1)2^{-n}$ if $l+j2^{-n}<Y_n(h)\leqq l+(j+1)2^{-n}$ and $Z_n(h)=-1+2^{-n}$ if $Y_n(h)=-1$ where $l=-1,0$ and $j=0,1,2,\cdots,2^n-1$. Then, for all $n\geqq 1$ and all $h\in H$,

$$\sum_{j=1}^{n} Y_j(h)\leqq\sum_{j=1}^{n} Z_j(h)\leqq\sum_{j=1}^{n} Y_j(h)+\sum_{j=1}^{n}2^{-j}<\sum_{j=1}^{n} Y_j(h)+1.$$

Hence

$$S(h)=\sup_{n\geqq 3}\left\{\sum_{j=1}^{n} Y_j(h)\Big/\sqrt{n\log\log n}\right\}$$
$$\leqq\sup_{n\geqq 3}\left\{\sum_{j=1}^{n} Z_j(h)\Big/\sqrt{n\log\log n}\right\}+1/\sqrt{3\log\log 3}.$$

Therefore $S\leqq T\leqq S+1/\sqrt{3\log\log 3}$. Now, suppose that $S\geqq 0$, then $S^2\leqq T^2$. If $S<0$, then $(T-1/\sqrt{3\log\log 3})^2\geqq S^2$. But $(T-1/\sqrt{3\log\log 3})^2\leqq 2(T^2+1/3\log\log 3)$. Therefore $S^2\leqq 2[T^2+1/3\log\log 3]$. By Lemma 4.4, $\nu(e^{tT^2})=\sigma(e^{tT^2})<\infty$ for all $0\leqq t<\infty$. Therefore $\sigma(e^{tS^2})<\infty$ for all $0<t<\infty$ (since e^{tS^2} is Borel-measurable).

Acknowledgements

I would like sincerely to thank Professors Lester E. Dubins and Willam D. Sudderth for their invaluable comments. I also want to thank the referee for his valuable suggestions and comments.

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